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POONEN'S QUESTION CONCERNING ISOGENIES BETWEEN SMART'S GENUS 2 CURVES

PAUL VAN WAMELEN

ABSTRACT. We describe a method for proving that two explicitly given genus two curves have isogenous jacobians. We apply the method to the list of genus 2 curves with good reduction away from 2 given by Smart. This answers a question of Poonen.

1. INTRODUCTION

By abuse of language we will say throughout that two curves are isogenous if their jacobians are isogenous.

In [5] Smart lists all genus 2 curves with good reduction away from 2. He organizes them into putative isogeny classes. In [4] Poonen asks for a proof that the isogeny classes as given by Smart are in fact correct. We solve this problem in a very concrete way by explicitly computing isogenies between the curves. This can be done by essentially the same method as was used to compute explicit CM morphisms in [6]. That is, compute such an isogeny numerically to high precision and then guess the exact values for the coefficients of this morphism. It can then be checked that these exact functions do define an isogeny. The morphism is computed numerically by going through the analytic representation of the jacobian of the curve—we compute the necessary integrals to go from the abelian variety to the torus, multiply by a matrix giving the complex representation of the morphism, and then use theta functions to go back to the abelian variety.

Note that this method, combined with comparing the number of points on reductions of the two curves (as done by Smart), should be able to decide whether two curves are isogenous or not. That is, by day compute the numbers of points on the two curves at bigger and bigger primes. If the number of points at some prime do not agree, the two curves are not isogenous. By night do the computations described in this paper to higher and higher precision. If the computation succeeds, the curves are isogenous.

I would like to thank Bjorn Poonen, who, upon hearing about the work in [6], pointed out that the same techniques should be tried on the present problem.

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2. Definitions and notation

Recall that any genus two curve C is hyperelliptic and can be given in one (or both) of the two forms

$$C : y^2 = f(x) = \begin{cases} \prod_{i=1}^{5}(x - a_i), \\ \prod_{i=1}^{6}(x - a_i). \end{cases} \tag{1}$$

where the a_i are distinct points in $\mathbb{C}$. We assume that the curve is defined over $\mathbb{Q}$, so $f(x) \in \mathbb{Q}[x]$.

If we regard C as a Riemann surface, the a_i are the branch points of the double cover of $\mathbb{P}$ by C. Let $\{A_1, A_2, B_1, B_2\}$ form a symplectic basis for the homology of C.

Let $\phi_1 = dx/y$ and $\phi_2 = xdx/y$. Then $\{\phi_1, \phi_2\}$ forms a basis for the holomorphic 1-forms on C ([2, p. 254] and [3, Proposition IIIa.5.2]), defined over $\mathbb{Q}$. We define the period matrix P of C by

$$P = \begin{pmatrix} \int_{B_1}\phi_1 & \int_{B_2}\phi_1 & \int_{A_1}\phi_1 & \int_{A_2}\phi_1 \\ \int_{B_1}\phi_2 & \int_{B_2}\phi_2 & \int_{A_1}\phi_2 & \int_{A_2}\phi_2 \end{pmatrix}. \tag{2}$$

Let ω_1 and ω_2 be the two 2×2 matrices such that $P = (\omega_1, \omega_2)$.

If we define τ to be the matrix $\omega_2^{-1}\omega_1$, then τ is in $\mathfrak{h}_2$, the Siegel upper half-space.

Let Λ be the free abelian group in $\mathbb{C}^2$ generated by the columns of P. Then Λ is a lattice in $\mathbb{C}^2$, and the jacobian J of C is given by $\mathbb{C}^2/\Lambda$.

The jacobian also has the structure of an abelian variety. Let α be an isogeny from one of these abelian varieties, J_1, to another, J_2. If we think of J_i as $\mathbb{C}^2/\Lambda_i$, then α induces a linear map from $\mathbb{C}^2$ to itself. We denote the 2×2 matrix giving this map by $\overline{\alpha}$. As $\overline{\alpha}$ represents a map from $\mathbb{C}^2/\Lambda_1$ to $\mathbb{C}^2/\Lambda_2$, there exists a 4×4 rational integer matrix M such that

$$\overline{\alpha}P_1 = P_2M. \tag{3}$$

As P_1 and P_2 are defined using holomorphic 1-forms defined over $\mathbb{Q}$, we have that $\overline{\alpha}$ is the complex representation of a morphism defined over a number field F if and only if $\overline{\alpha}$ has entries in F. So if we are looking for a rational isogeny, we want $M \in \mathrm{M}_4(\mathbb{Z})$ and $\overline{\alpha} \in \mathrm{M}_2(\mathbb{Q})$.

Recall that the jacobian is the unique abelian variety birationally equivalent to the symmetric product of the curve with itself, $C^{(2)}$. We therefore think of points on the jacobian as unordered pairs of points on the curve (written as a sum of two points). The only pairs for which this breaks down are pairs $Q + \iota(Q)$, where ι is the hyperelliptic involution. All such points correspond to the zero element of the jacobian. For the rest of this paper we will denote by $Q_1 + Q_2$ the image of $Q + Q_0$ under the map induced by α from $C_1^{(2)}$ to $C_2^{(2)}$, where $Q = (x, y)$, $Q_i = (x_i, y_i)$, $i = 1, 2$, and Q_0 is some fixed point on the curve C_1. We will always pick Q_0 to be one of the Weierstrass points. That is, if $f(x)$ (in the hyperelliptic equation (1)) is a sextic we let $Q_0 = (a_i, 0)$ for some i. If $f(x)$ is a quintic we will pick Q_0 to be the (rational) point at infinity.

Note that $x_1 + x_2$ and x_1x_2 can be considered as meromorphic functions on the curve. As these functions do not depend on the y-coordinate of Q, we see that $x_1 + x_2$ and x_1x_2 are rational functions in x. From now on $s_1 = x_1 + x_2$ and $s_2 = x_1x_2$ will denote these rational functions of x. If α is defined over $\mathbb{Q}$, we see that the coefficients of s_1 and s_2 as functions of x are in $\mathbb{Q}(a_i)$ (where $Q_0 = (a_i, 0)$).

We will now proceed as follows. Given two genus 2 curves C_1 and C_2, we can compute their period matrices P_1 and P_2 to high precision by numerically doing the integrals in (2). In Section 3 we will show how, from P_1 and P_2, we can find matrices $\overline{\alpha}$ and M such that (3) holds. From $\overline{\alpha}$ we can compute approximations to the rational functions giving α as a morphism of abelian varieties. See Section 4. We then guess exact values for these rational functions and check that they do define an isogeny (Section 5).

To compute the period matrix of a curve C we actually used the following method. If the curve is given by a sextic we find a linear fractional transformation that sends Q_0 to ∞. This transforms the curve into one given by a quintic (but not defined over $\mathbb{Q}$). Now we use the same method (and programs) used in [6] to compute the period matrix P of this curve to high precision. For the analytic part of our computations this form of the period matrix is sufficient and, in fact, simplifies things. On the other hand, for finding $\overline{\alpha}$ in the next section we certainly need the correct period matrix for the original curve in sextic form (so that it's defined over $\mathbb{Q}$). This can be found by multiplying P by the 2×2 jacobian matrix coming from the linear fractional change of variables.

3. Finding a matrix representing a rational isogeny

Given two period matrices P_1 and P_2 to high precision, we want to find $\overline{\alpha} \in \mathrm{M}_2(\mathbb{Q})$ and $M \in \mathrm{M}_4(\mathbb{Z})$ such that (3) holds.

Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, where

$$a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad b = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, \quad c = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}, \quad d = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix};$$

then we see that equation (3) becomes

$$(\tau_2 a + c) = (\tau_2 b + d)\tau_1. \tag{4}$$

Let

$$\tau_1 = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \quad \text{and} \quad \tau_2 = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}.$$

Also let

$$\begin{aligned}
\mathbf{v}_1 &= \{1, s_{11}, s_{12}, t_{11}, s_{11}t_{11}, s_{12}t_{11}, t_{21}, s_{11}t_{21}, s_{12}t_{21}\},\\
\mathbf{v}_2 &= \{1, s_{11}, s_{12}, t_{12}, s_{11}t_{12}, s_{12}t_{12}, t_{22}, s_{11}t_{22}, s_{12}t_{22}\},\\
\mathbf{v}_3 &= \{1, s_{21}, s_{22}, t_{11}, s_{21}t_{11}, s_{22}t_{11}, t_{21}, s_{21}t_{21}, s_{22}t_{21}\},\\
\mathbf{v}_4 &= \{1, s_{21}, s_{22}, t_{12}, s_{21}t_{12}, s_{22}t_{12}, t_{22}, s_{21}t_{22}, s_{22}t_{22}\},
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{a}_1 &= \{-c_{11}, -a_{11}, -a_{21}, d_{11}, b_{11}, b_{21}, d_{12}, b_{12}, b_{22}\},\\
\mathbf{a}_2 &= \{-c_{12}, -a_{12}, -a_{22}, d_{11}, b_{11}, b_{21}, d_{12}, b_{12}, b_{22}\},\\
\mathbf{a}_3 &= \{-c_{21}, -a_{11}, -a_{21}, d_{21}, b_{11}, b_{21}, d_{22}, b_{12}, b_{22}\},\\
\mathbf{a}_4 &= \{-c_{22}, -a_{12}, -a_{22}, d_{21}, b_{11}, b_{21}, d_{22}, b_{12}, b_{22}\}.
\end{aligned}$$

In this notation equation (4) is

$$\mathbf{v}_i \cdot \mathbf{a}_i = 0, \text{ for } i = 1, 2, 3, 4.$$

Given τ_1 and τ_2 to high precision, we can use an LLL based algorithm to find integer linear dependencies among the entries of the $\mathbf{v}_i$, that is, vectors $\mathbf{l}$, with integer entries, such that $\mathbf{v}_i \cdot \mathbf{l} = 0$ (see for instance [1, Algorithm 2.7.4]). In fact, for each of the four $\mathbf{v}_i$, we compute a basis $\{\mathbf{l}_{ij}\}_{j=1}^{n_i}$ for all the linear dependencies that the algorithm finds. So for integer valued variables k_{ij} we get

$$\mathbf{a}_i = \sum_j^{n_i} k_{ij}\mathbf{l}_{ij} \quad \text{for } i = 1, 2, 3, 4.$$

We now solve this linear system for the entries of M with the k_{ij} in $\mathbb{Z}$ (Maple's `isolve` worked well). This gives new integer valued parameters k_i, and we can write

$$M = \sum_i^n k_i M_i. \tag{5}$$

Here each M_i is an integer matrix corresponding to an isogeny, not necessarily defined over $\mathbb{Q}$. For each M_i we can compute the corresponding $\overline{\alpha}_i$. If $P_1 = (\omega_1, \omega_2)$ and $P_2 = (\pi_1, \pi_2)$, then

$$\overline{\alpha}_i = \omega_1^{-1}(\pi_1 a + \pi_2 c).$$

The entries of $\overline{\alpha}_i$ will be (approximations to) algebraic integers in the field of definition of the isogeny corresponding to M_i. If we computed them to high enough precision, we can recognize them as exact algebraic numbers by an algorithm, again based on some form of lattice reduction (see for instance [1, Section 2.7.2]). If we embed all the exact $\overline{\alpha}_i$'s into a common number field, it becomes another diophantine linear algebra problem to find integer coefficients k_i such that $\overline{\alpha} = \sum_i^n k_i \alpha_i$ is in $\mathrm{M}_2(\mathbb{Q})$. In this way we find (hopefully) all $\overline{\alpha} \in \mathrm{M}_2(\mathbb{Q})$ and the corresponding $M \in \mathrm{M}_4(\mathbb{Z})$ that solve equation (3). The degree of an isogeny is given by $\det(M)$, so if we want to have small s_1 and s_2 we should look for an $\overline{\alpha}$ with $\det(M)$ as small as possible. In our case we just examined determinants of the M corresponding to small values of the parameters.

4. Guessing s_1 and s_2

Just as in [6], we will guess exact values of $s_1 = x_1 + x_2$ and $s_2 = x_1 x_2$ by computing the values of x_1 and x_2 at enough points x so that we can solve a linear system in order to find approximations for the coefficients of the rational functions s_1 and s_2. So we take many points Q, move them to the corresponding quintic if necessary, and apply the method in [6] to find the (approximate) images on the second curve (where multiplication by $\overline{\alpha}$ now takes points on one jacobian to another). Of course, the image points might need to be moved back to a curve in sextic form. It is here that our work is simplified by working with the period matrix for the curve in quintic form. The theory in [3] for computing the maps from the algebraic jacobian to the analytic jacobian and back (by theta functions) is worked out in detail only for the quintic case.

It turned out that the functions s_1 and s_2 are relatively simple for all of Smart's curves (see Section 6), and working with a precision of about 100 was more than adequate and could be done in a very reasonable time.

5. Proving that s_1 and s_2 are correct

So assume that we have an exact $\overline{\alpha}$ and (guesses for) s_1 and s_2 as exact rational functions of x. We want to check whether this data represents an isogeny.

Note that $(y_1y_2)^2 = f_2(x_1)f_2(x_2)$ can be written as a function of s_1 and s_2, and (by finding it's square root) we can therefore compute y_1y_2 up to a sign as a rational function q_1 of x,

$$y_1y_2 = q_1(x).$$

Note that we can write $y_1 + y_2 = yq_2$ and $y_1x_1 + y_2x_2 = yq_3$, where q_2 and q_3 are rational functions of x. Now

$$(y_1 + y_2)^2 = f_2(x_1) + 2q_1 + f_2(x_2) = f_1(x)r_2(s_1, s_2),$$
$$(x_1y_1 + x_2y_2)^2 = x_1^2 f_2(x_1) + 2x_1x_2q_1 + x_2^2 f_2(x_2) = f_1(x)r_3(s_1, s_2),$$

so that we can also find $y_1 + y_2$ and $x_1y_1 + x_2y_2$, up to two more sign ambiguities, as y times rational functions of x, namely the square roots of r_2 and r_3.

As in [6], we want to check that

$$\frac{1}{y_1}\frac{dx_1}{dx} + \frac{1}{y_2}\frac{dx_2}{dx} = \frac{\alpha_{11} + \alpha_{12}x}{y}$$
$$\frac{x_1}{y_1}\frac{dx_1}{dx} + \frac{x_2}{y_2}\frac{dx_2}{dx} = \frac{\alpha_{21} + \alpha_{22}x}{y},$$

where $\overline{\alpha} = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix}$. If we let $'$ denote differentiation with respect to x we can verify that

$$\left(\frac{1}{y_1}\frac{dx_1}{dx} + \frac{1}{y_2}\frac{dx_2}{dx}\right) y = f_1(x) \left(\frac{s_1^2 s_1' q_2 - 2s_1' s_2 q_2 - s_1 s_2' q_2 - s_1 s_1' q_3 + 2s_2' q_3}{(s_1^2 - 4s_2)q_1}\right)$$

and

$$\left(\frac{x_1}{y_1}\frac{dx_1}{dx} + \frac{x_2}{y_2}\frac{dx_2}{dx}\right) y$$
$$= f_1(x) \left(\frac{s_1^3 s_1' q_2 - 3s_1 s_1' s_2 q_2 - s_1^2 s_2' q_2 + 2s_2 s_2' q_2 - s_1^2 s_1' q_3 + 2s_1' s_2 q_3 + s_1 s_2' q_3}{(s_1^2 - 4s_2)q_1}\right).$$

We can evaluate the right sides of these two equations in terms of x and check for which choice of the unknown signs we get linear polynomials and the correct coefficients α_{ij}. In this way we not only check that the given values of s_1 and s_2 do give a morphism with complex representation $\overline{\alpha}$, but we also get the correct signs for $y_1 + y_2$ and $x_1y_1 + x_2y_2$.

So we see that having $x_1 + x_2$, x_1x_2 and $\overline{\alpha}$ makes it easy to work out the image (in $C_2^{(2)}$) of $Q + Q_0$ under the isogeny. If we want to know the image of a general element of J_1, say given by $R_1 + R_2$, with R_1 and R_2 points on C_1, we compute the images of $R_1 + Q_0$ and $R_2 + Q_0$ and add them on J_2. This will be the image of $R_1 + R_2$, because an isogeny is a group homomorphism and Q_0 is a Weierstrass point, so that $Q_0 + Q_0$ represents the zero element on J_1.

6. Results

The full results, including a Mathematica program for checking the data (as explained in section 5), are available at `http://math.lsu.edu/~wamelen/`.

We use the following numbering system. The isogeny classes are numbered as in [5, Table 7]. The curves in each class are numbered by the order of occurrence in the class in [5, Table 7].

TABLE 1. The isomorphic pairs

n	i	j	n	i	j	n	i	j	n	i	j	n	i	j	n	i	j	n	i	j
6	1	2	11	1	2	22	1	2	108	1	2	127	3	4	144	1	2	154	3	4
6	4	5	11	3	4	45	2	3	109	1	2	128	1	2	144	3	4	155	1	2
7	1	2	11	5	6	46	2	3	110	2	3	129	1	2	145	1	2	155	3	4
7	4	5	16	2	3	100	2	3	110	4	5	132	2	3	145	3	4	160	1	2
8	1	2	17	2	3	101	2	3	111	2	3	133	2	3	146	1	2	161	1	2
9	1	2	17	4	5	106	1	2	111	4	5	134	2	3	147	1	2	162	1	2
10	1	2	18	2	3	106	3	4	126	1	2	135	2	3	148	1	2	163	1	2
10	3	4	18	4	5	107	1	2	126	3	4	136	1	2	149	1	2			
10	5	6	21	1	2	107	3	4	127	1	2	137	1	2	154	1	2			

Some of the curves in Smart's table are not only isogenous over $\mathbb{Q}$ but isomorphic. These pairs of curves are listed in Table 1. The actual isomorphisms can easily be found by the methods in [5, Section 7]. (Note that the corresponding sextic forms for these isomorphic curves are *in*equivalent.)

In Table 3 we list $s_1 = x_1 + x_2$, $s_2 = x_1 x_2$ and $\overline{\alpha}$ for enough pairs of curves to give a connected graph on all the curves in each isogeny class. As was pointed out in the previous section, the given data makes it easy to compute the given isogeny algebraically.

Looking at the table of curves, one quickly notices that many of the polynomials are related by replacing x with a small multiple of x and possibly multiplying the whole polynomial by a small integer. This means that many of the functions (s_1 and s_2) in our table are similarly related. We therefore first list all the functions we need (in Table 2) and then express the isogenies in terms of these functions in Table 3. This substantially reduces the length of the tables.

Note that all the entries in Table 3 are defined over $\mathbb{Q}$. This need not have been the case. For some of the curves there exists no rational Weierstrass point, and we must use a non-rational point Q_0. This usually means that s_1 and s_2 are only defined over the field over which Q_0 is defined. To minimize the size of s_1 and s_2 we computed them for several choices of Q_0, and it turned out that in all cases we were lucky and were able to find a choice of Q_0 that gives s_1 and s_2 in $\mathbb{Q}(x)$.

TABLE 2. The functions

k	f_k	g_k
1	$\frac{8}{(-2+x)x}$	$\frac{4}{(-2+x)^2}$
2	$\frac{-32}{8+x^2}$	$\frac{4}{x^2}$
3	$\frac{-2}{x}$	$\frac{2(1+x^2)}{x^2}$
4	$\frac{-2x}{2+x^2}$	$\frac{2}{2+x^2}$
5	$\frac{2(-2+x)}{2+x}$	$\frac{4-12x+x^2}{(2+x)^2}$
6	$\frac{8(2+3x)}{4+8x+x^2}$	$\frac{8(2+3x)}{4+8x+x^2}$
7	$\frac{-16(3+x)}{-7+x^2}$	$\frac{-2(37+24x+5x^2)}{-7+x^2}$
8	$\frac{8x}{6-4x+3x^2}$	$\frac{-2(2-4x+x^2)}{6-4x+3x^2}$
9	$\frac{2x^2}{(2+x)^2}$	$\frac{x^2}{(-2+x)^2}$
10	$\frac{-2(-2+x^2)}{-2-4x+x^2}$	$\frac{2-4x-x^2}{-2-4x+x^2}$
11	$\frac{-4(2-4x+x^2)}{6-4x+x^2}$	$\frac{2-4x+3x^2}{6-4x+x^2}$
12	$\frac{2(-5+2x+x^2)}{3+2x+x^2}$	$\frac{9-10x+3x^2}{3+2x+x^2}$
13	$\frac{-2(-4+x^2)}{4+12x+x^2}$	$\frac{4-4x+x^2}{4+12x+x^2}$
14	$\frac{-2(8-8x+x^2)}{-8+x^2}$	$\frac{32-48x+36x^2-10x^3+x^4}{2(16-8x-2x^2+x^3)}$
15	$\frac{-2(8-4x+x^2)}{-8+x^2}$	$\frac{128-144x+72x^2-14x^3+x^4}{4(32-8x-4x^2+x^3)}$
16	$\frac{-4(2+4x+3x^2+2x^3)}{2+4x+5x^2+4x^3}$	$\frac{2(2+8x+5x^2+2x^3)}{2+4x+5x^2+4x^3}$
17	$\frac{2x}{-1-x^2+x^4}$	$\frac{2(-1-3x^2+2x^4)}{-1-x^2+x^4}$
18	$\frac{-4x}{-4+2x^2+x^4}$	$\frac{-2(-2+3x^2+x^4)}{-4+2x^2+x^4}$
19	$\frac{-4(2+6x+5x^2+x^3)}{8+12x+6x^2+2x^3+x^4}$	$\frac{8+20x+26x^2+14x^3+3x^4}{8+12x+6x^2+2x^3+x^4}$
20	$\frac{2(16+8x^2+x^4)}{-16+16x-12x^3+x^4}$	$\frac{-16-48x+4x^3+x^4}{-16+16x-12x^3+x^4}$
21	$\frac{-2(-12-24x-20x^2-4x^3+x^4)}{20+24x+12x^2+4x^3+x^4}$	$\frac{12+40x+52x^2+28x^3+7x^4}{20+24x+12x^2+4x^3+x^4}$
22	$\frac{2(-64-256x+112x^2+256x^3-28x^4-16x^5+x^6)}{(-2+x)^2(16+32x+40x^2-8x^3+x^4)}$	$\frac{4-4x+x^2}{(2+x)^2}$
23	$\frac{32-128x+152x^2-72x^3+17x^4-2x^5}{(-2+x)^4x}$	$\frac{32-64x+40x^2-8x^3+x^4}{(-2+x)^4}$

TABLE 2. (continued)

k	f_k	g_k
24	$\frac{-4(64+192x+240x^2+144x^3+35x^4+x^5)}{x(-8-4x+x^2)^2}$	$\frac{4(-64-128x-80x^2-16x^3+x^4)}{(-8-4x+x^2)^2}$
25	$\frac{2(2x-4x^3+x^5)}{-2-4x^2+3x^4}$	$\frac{-8+2x^2+x^6}{-2-4x^2+3x^4}$
26	$\frac{-4(64-64x+48x^2-24x^3+5x^4)}{-128+192x-160x^2+64x^3-18x^4+x^5}$	$\frac{-256+384x-640x^2+320x^3-84x^4+10x^5-x^6}{2(-128+192x-160x^2+64x^3-18x^4+x^5)}$
27	$\frac{-4(4-8x+4x^2+4x^3+x^4)}{-8+36x-16x^2-12x^3+2x^4+x^5}$	$\frac{4(4-4x-12x^2+8x^3+9x^4+x^5)}{x(-8+36x-16x^2-12x^3+2x^4+x^5)}$
28	$\frac{4(4-16x+4x^2+8x^3+x^4)}{-16+36x-32x^2-12x^3+4x^4+x^5}$	$\frac{4(4-8x-12x^2+16x^3+9x^4+2x^5)}{x(-16+36x-32x^2-12x^3+4x^4+x^5)}$
29	$\frac{2(4x+8x^2+4x^3-4x^4+x^5)}{-4-4x+12x^2+8x^3-9x^4+x^5}$	$\frac{-8x-36x^2-16x^3+12x^4+2x^5-x^6}{-4-4x+12x^2+8x^3-9x^4+x^5}$
30	$\frac{-2(4x+16x^2+4x^3-8x^4+x^5)}{-4-8x+12x^2+16x^3-9x^4+2x^5}$	$\frac{-16x-36x^2-32x^3+12x^4+4x^5-x^6}{-4-8x+12x^2+16x^3-9x^4+2x^5}$
31	$\frac{-2(-4+20x-28x^2+4x^3-x^4+x^5)}{-4+24x-44x^2+32x^3-9x^4+2x^5}$	$\frac{-4+32x-48x^2+40x^3-37x^4+12x^5-x^6}{-4+24x-44x^2+32x^3-9x^4+2x^5}$
32	$\frac{-2(4+16x+20x^2+12x^3+5x^4+2x^5)}{4+20x+28x^2+24x^3+9x^4+3x^5}$	$\frac{4+36x+48x^2+48x^3+29x^4+11x^5+x^6}{4+20x+28x^2+24x^3+9x^4+3x^5}$
33	$\frac{-2(-256+448x-256x^2+128x^3-28x^4+5x^5)}{-256+576x-320x^2+160x^3-36x^4+7x^5}$	$\frac{-256+1216x-768x^2+416x^3-100x^4+21x^5-x^6}{-256+576x-320x^2+160x^3-36x^4+7x^5}$
34	$\frac{2(-9+34x-36x^2+8x^3-4x^4+8x^5)}{19-76x+108x^2-48x^3-20x^4+16x^5}$	$\frac{19-40x+8x^2+32x^3-36x^4+32x^5-16x^6}{19-76x+108x^2-48x^3-20x^4+16x^5}$
35	$\frac{4(-4x+4x^3+x^5)}{-16-4x^2+x^6}$	$\frac{4(4+4x^2+3x^4)}{-16-4x^2+x^6}$
36	$\frac{-2(-64+128x+48x^2-12x^4-8x^5+x^6)}{64-64x+368x^2-96x^3+92x^4-4x^5+x^6}$	$\frac{64+64x+112x^2-160x^3+28x^4+4x^5+x^6}{64-64x+368x^2-96x^3+92x^4-4x^5+x^6}$
37	$\frac{2(-21-28x-25x^2-8x^3-3x^4+4x^5+x^6)}{1-26x-21x^2-28x^3-5x^4-2x^5+x^6}$	$\frac{21+50x+71x^2+60x^3+39x^4+10x^5+5x^6}{1-26x-21x^2-28x^3-5x^4-2x^5+x^6}$
38	$\frac{-2(-64-128x+432x^2-108x^4+8x^5+x^6)}{64-704x-16x^2-672x^3-4x^4-44x^5+x^6}$	$\frac{64-832x-272x^2+160x^3-68x^4-52x^5+x^6}{64-704x-16x^2-672x^3-4x^4-44x^5+x^6}$
39	$\frac{2(-4-12x-8x^2+12x^3+17x^4+7x^5+x^6)}{4-8x+32x^3+31x^4+10x^5+x^6}$	$\frac{4+16x+12x^2-8x^3-5x^4+4x^5+2x^6}{4-8x+32x^3+31x^4+10x^5+x^6}$
40	$\frac{-4(-2x+x^5)}{4+2x^2-2x^4+x^6}$	$\frac{2(-4+4x^2+12x^4+4x^6-x^8+x^{10})}{8+20x^2+8x^4-4x^6+2x^8+x^{10}}$

TABLE 2. (continued)

k	f_k	g_k
41	$\frac{-4(-64+384x-80x^2+20x^4-24x^5+x^6)}{64+960x-784x^2+800x^3-196x^4+60x^5+x^6}$	$\frac{4(1024-4096x+40192x^2-32768x^3+30848x^4-7680x^5+7712x^6-2048x^7+628x^8-16x^9+x^{10})}{1024+17408x+11520x^2-111616x^3+111744x^4-93824x^5+27936x^6-6976x^7+180x^8+68x^9+x^{10}}$
42	$\frac{-4(4-24x^2+48x^4-36x^6+7x^8)}{x(-12+48x^2-60x^4+24x^6+x^8)}$	$\frac{2(-4+24x^2-60x^4+72x^6-33x^8+6x^{10})}{x^2(-12+48x^2-60x^4+24x^6+x^8)}$
43	$\frac{8(x-6x^5+x^9)}{-3-19x^2-42x^4-10x^6+x^8+x^{10}}$	$\frac{-2(-1+5x^2+26x^4+30x^6+3x^8+x^{10})}{-3-19x^2-42x^4-10x^6+x^8+x^{10}}$
44	$\frac{4(6+9x+10x^2+8x^3+4x^4+18x^5+12x^6+8x^7-2x^8-7x^9+2x^{10})}{4+x+4x^2+8x^3+8x^4+18x^5+8x^6+8x^7+4x^8+x^9+4x^{10}}$	$\frac{2(18+39x+50x^2+24x^3-12x^4-2x^5+20x^6+24x^7+2x^8-25x^9+2x^{10})}{4+x+4x^2+8x^3+8x^4+18x^5+8x^6+8x^7+4x^8+x^9+4x^{10}}$
45	$\frac{-8}{(2+x)^2}$	$\frac{x^2}{4}$
46	$\frac{-8x}{-2+x^2}$	$\frac{16+8x+x^2}{16}$
47	$\frac{-4(1+x^2)}{-1-2x+x^2}$	$\frac{8(2+x)}{x(4+x)}$
48	$\frac{-2(-1+2x^2)}{3-4x+2x^2}$	$\frac{4(4+3x)}{x(2+x)}$
49	$\frac{-2(8+8x-24x^2-16x^3+36x^4-14x^5+x^6)}{(-2+x)^3x(-2+x^2)}$	$\frac{-16+6x-x^2}{-4+x}$
50	$\frac{4(-1-8x+7x^2+32x^3-7x^4-8x^5+x^6)}{(1+x)^2(1+4x+10x^2-4x^3+x^4)}$	$\frac{-2(-3x+2x^2)}{-4+3x}$
51	$\frac{-2(-4096+7168x-768x^2-800x^3+24x^4+20x^5+x^6)}{x^3(32-8x-4x^2+x^3)}$	$\frac{4-8x+x^2}{(-2+x)x}$
52	0	$\frac{64+16x+x^2}{x^2}$

TABLE 3. The isogenies

n	i	j	$\overline{\alpha}$	s_1	s_2
2	1	2	$\begin{pmatrix} 0 & -1 \\ -2 & 1 \end{pmatrix}$	$f_{49}(x)$	$g_9(x)$
2	1	3	$\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ 1 & 0 \end{pmatrix}$	$f_1(2x)$	$4g_1(2x)$
2	1	4	$\begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$	0	$g_{50}(x)$
3	1	2	$\begin{pmatrix} 0 & \frac{1}{2} \\ 2 & 0 \end{pmatrix}$	0	$4g_2(x)$
3	1	3	$\begin{pmatrix} 2 & \frac{1}{2} \\ 0 & -\frac{1}{2} \end{pmatrix}$	$f_{51}(2x)$	$g_{52}(2x)$
3	1	4	$\begin{pmatrix} -2 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}$	$-f_{51}(-2x)$	$g_{52}(-2x)$
4	1	2	$\begin{pmatrix} 0 & 2 \\ 8 & 0 \end{pmatrix}$	0	$4g_2(x)$
4	1	3	$\begin{pmatrix} 2 & \frac{1}{4} \\ 0 & -\frac{1}{2} \end{pmatrix}$	$2f_{51}(x)$	$4g_{52}(x)$
4	1	4	$\begin{pmatrix} -2 & \frac{1}{4} \\ 0 & \frac{1}{2} \end{pmatrix}$	$-2f_{51}(-x)$	$4g_{52}(-x)$
6	1	4	$\begin{pmatrix} 2 & 0 \\ -2 & 1 \end{pmatrix}$	$f_{33}(x)$	$g_{33}(x)$
6	2	3	$\begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}$	$f_2(x)$	$g_2(x)$
7	1	3	$\begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}$	$-f_2(x)$	$g_2(x)$
7	1	4	$\begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$	$f_{33}(-2x)$	$g_{33}(-2x)$
8	1	5	$\begin{pmatrix} -\frac{1}{2} & 0 \\ -1 & \frac{1}{2} \end{pmatrix}$	4	$g_{49}(2x)$
8	3	5	$\begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$	2	$g_{51}(x)$
8	4	5	$\begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -1 & 0 \end{pmatrix}$	$f_6(2x)$	$g_6(2x)$
9	1	5	$\begin{pmatrix} -2 & 0 \\ -4 & 1 \end{pmatrix}$	4	$g_{49}(x)$
9	3	5	$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$	2	$g_{51}(2x)$
9	4	5	$\begin{pmatrix} -2 & -1 \\ -4 & 0 \end{pmatrix}$	$f_6(x)$	$g_6(x)$
10	1	3	$\begin{pmatrix} 2 & 0 \\ 2 & -\frac{1}{2} \end{pmatrix}$	$f_{15}(x)$	$g_{15}(x)$
10	1	5	$\begin{pmatrix} 2 & 0 \\ 2 & -1 \end{pmatrix}$	$f_{14}(x)$	$g_{14}(x)$
11	1	3	$\begin{pmatrix} 1 & 0 \\ 1 & \frac{1}{2} \end{pmatrix}$	$f_{15}(-2x)$	$g_{15}(-2x)$
11	1	5	$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	$f_{14}(-2x)$	$g_{14}(-2x)$
16	1	2	$\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}$	$f_{16}(x)$	$g_{16}(x)$
17	1	2	$\begin{pmatrix} 0 & -\frac{1}{2} \\ 4 & 0 \end{pmatrix}$	$4f_2(x)$	$16g_2(x)$
17	1	5	$\begin{pmatrix} -2 & 0 \\ -2 & 1 \end{pmatrix}$	$f_{26}(x)$	$g_{26}(x)$
18	1	2	$\begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}$	$-2f_2(x)$	$g_2(x)$
18	1	5	$\begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}$	$f_{26}(-2x)$	$g_{26}(-2x)$
19	1	2	$\begin{pmatrix} 0 & 2 \\ 4 & 0 \end{pmatrix}$	0	$g_2(x)$
19	1	3	$\begin{pmatrix} 2 & -1 \\ -2 & -1 \end{pmatrix}$	$f_{20}(x)$	$g_{20}(x)$
20	1	2	$\begin{pmatrix} 0 & \frac{1}{2} \\ 1 & 0 \end{pmatrix}$	0	$g_2(x)$
20	1	3	$\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$	$f_{20}(2x)$	$g_{20}(2x)$
21	1	3	$\begin{pmatrix} 0 & \frac{1}{2} \\ 4 & -1 \end{pmatrix}$	-4	$4g_{52}(-2x)$
21	2	4	$\begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$	2	$g_{52}(4x)$
21	2	5	$\begin{pmatrix} 0 & \frac{1}{2} \\ 1 & 0 \end{pmatrix}$	0	$g_{47}(2x)$
22	1	3	$\begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix}$	-2	$g_{52}(-4x)$
22	2	4	$\begin{pmatrix} 0 & \frac{1}{2} \\ 4 & 1 \end{pmatrix}$	4	$4g_{52}(2x)$
22	2	5	$\begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}$	0	$g_{47}(x)$
23	1	2	$\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$	0	$g_2(x)$
24	1	2	$\begin{pmatrix} -4 & -2 \\ 4 & 0 \end{pmatrix}$	$f_{45}(-x)$	$g_1(-x)$
24	1	3	$\begin{pmatrix} 0 & -2 \\ 8 & 2 \end{pmatrix}$	-2	$g_{52}(2x)$
24	1	4	$\begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}$	0	$g_{47}(x)$
25	1	2	$\begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ 1 & 0 \end{pmatrix}$	$2f_{45}(-2x)$	$4g_1(-2x)$
25	1	3	$\begin{pmatrix} 0 & -\frac{1}{2} \\ 2 & 1 \end{pmatrix}$	-4	$4g_{52}(4x)$
25	1	4	$\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$	0	$g_{47}(2x)$
26	1	2	$\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ 1 & 0 \end{pmatrix}$	$-2f_{45}(2x)$	$4g_1(2x)$
26	1	3	$\begin{pmatrix} 0 & -\frac{1}{2} \\ 2 & -1 \end{pmatrix}$	4	$4g_{52}(-4x)$
26	1	4	$\begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}$	0	$g_{47}(-2x)$
27	1	2	$\begin{pmatrix} 4 & -2 \\ 4 & 0 \end{pmatrix}$	$-f_{45}(x)$	$g_1(x)$
27	1	3	$\begin{pmatrix} 0 & -2 \\ 8 & -2 \end{pmatrix}$	2	$g_{52}(-2x)$
27	1	4	$\begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix}$	0	$g_{47}(-x)$
28	1	2	$\begin{pmatrix} 0 & \frac{1}{2} \\ 1 & 0 \end{pmatrix}$	0	$g_2(x)$
28	1	3	$\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} \end{pmatrix}$	$f_{44}(x)$	$g_{44}(x)$
28	1	4	$\begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{3}{2} & -\frac{1}{2} \end{pmatrix}$	$f_{44}(-x)$	$g_{44}(-x)$
29	1	2	$\begin{pmatrix} 4 & 0 \\ -4 & -2 \end{pmatrix}$	-2	$g_{46}(2x)$
30	1	2	$\begin{pmatrix} \frac{1}{2} & 0 \\ -1 & -1 \end{pmatrix}$	-4	$4g_{46}(4x)$

TABLE 3. (continued)

n	i	j	$\overline{\alpha}$	s_1	s_2
31	1	2	$\begin{pmatrix} \frac{1}{2} & 0 \\ 1 & -1 \end{pmatrix}$	4	$4g_{46}(-4x)$
32	1	2	$\begin{pmatrix} 4 & 0 \\ 4 & -2 \end{pmatrix}$	2	$g_{46}(-2x)$
33	1	2	$\begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -1 \end{pmatrix}$	0	$16g_{45}(x)$
34	1	2	$\begin{pmatrix} 4 & 2 \\ 0 & 2 \end{pmatrix}$	$f_9(-x)$	$g_9(-x)$
35	1	2	$\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{pmatrix}$	$2f_9(-2x)$	$4g_9(-2x)$
36	1	2	$\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ 0 & -1 \end{pmatrix}$	$-2f_9(2x)$	$4g_9(2x)$
37	1	2	$\begin{pmatrix} -4 & 2 \\ 0 & -2 \end{pmatrix}$	$-f_9(x)$	$g_9(x)$
38	1	2	$\begin{pmatrix} -1 & -1 \\ -2 & 2 \end{pmatrix}$	$f_{41}(2x)$	$g_{41}(2x)$
39	1	2	$\begin{pmatrix} -4 & -2 \\ -8 & 4 \end{pmatrix}$	$f_{41}(x)$	$g_{41}(x)$
40	1	2	$\begin{pmatrix} -4 & 2 \\ -8 & -4 \end{pmatrix}$	$f_{41}(-x)$	$g_{41}(-x)$
41	1	2	$\begin{pmatrix} 1 & -1 \\ -2 & -2 \end{pmatrix}$	$-f_{41}(-2x)$	$g_{41}(-2x)$
42	1	2	$\begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -1 \end{pmatrix}$	0	$16g_{45}(x)$
43	1	2	$\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ 1 & 1 \end{pmatrix}$	$f_{50}(2x)$	$4g_{22}(2x)$
44	1	2	$\begin{pmatrix} 4 & 2 \\ -4 & 2 \end{pmatrix}$	$f_{22}(-x)$	$g_{22}(-x)$
45	1	3	$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$	$f_{31}(x)$	$g_{31}(x)$
46	1	2	$\begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$	$f_{31}(-x)$	$g_{31}(-x)$
47	1	2	$\begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -1 \end{pmatrix}$	0	$16g_{45}(x)$
48	1	2	$\begin{pmatrix} -4 & 0 \\ 0 & -2 \end{pmatrix}$	0	$g_{45}(x)$
49	1	2	$\begin{pmatrix} 2 & -1 \\ 4 & 0 \end{pmatrix}$	$-2f_{45}(x)$	$4g_1(x)$
50	1	2	$\begin{pmatrix} \frac{1}{4} & -\frac{1}{4} \\ 1 & 0 \end{pmatrix}$	$-4f_{45}(2x)$	$16g_1(2x)$
51	1	2	$\begin{pmatrix} -\frac{1}{4} & -\frac{1}{4} \\ 1 & 0 \end{pmatrix}$	$4f_{45}(-2x)$	$16g_1(-2x)$
52	1	2	$\begin{pmatrix} -2 & -1 \\ 4 & 0 \end{pmatrix}$	$2f_{45}(-x)$	$4g_1(-x)$
53	1	2	$\begin{pmatrix} -2 & 2 \\ -2 & -2 \end{pmatrix}$	$f_{38}(2x)$	$g_{38}(2x)$
54	1	2	$\begin{pmatrix} -4 & 2 \\ -4 & -2 \end{pmatrix}$	$f_{38}(x)$	$g_{38}(x)$
55	1	2	$\begin{pmatrix} -4 & -2 \\ -4 & 2 \end{pmatrix}$	$f_{38}(-x)$	$g_{38}(-x)$
56	1	2	$\begin{pmatrix} -2 & -2 \\ -2 & 2 \end{pmatrix}$	$f_{38}(-2x)$	$g_{38}(-2x)$
57	1	2	$\begin{pmatrix} 4 & 0 \\ -8 & -2 \end{pmatrix}$	-4	$4g_{46}(x)$
58	1	2	$\begin{pmatrix} \frac{1}{2} & 0 \\ 2 & -1 \end{pmatrix}$	8	$16g_{46}(-2x)$
63	1	2	$\begin{pmatrix} -4 & -2 \\ -4 & 2 \end{pmatrix}$	$f_{13}(x)$	$g_{13}(x)$
63	2	3	$\begin{pmatrix} 0 & 2 \\ 4 & 0 \end{pmatrix}$	$f_{42}(x)$	$g_{42}(x)$
63	3	4	$\begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}$	$f_4(x)$	$g_4(x)$
64	1	2	$\begin{pmatrix} 2 & 2 \\ -2 & 2 \end{pmatrix}$	$-f_{13}(2x)$	$g_{13}(2x)$
64	2	3	$\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$	$f_{42}(x)$	$g_{42}(x)$
64	3	4	$\begin{pmatrix} 0 & -2 \\ 4 & 0 \end{pmatrix}$	$f_4(x)$	$g_4(x)$
65	1	2	$\begin{pmatrix} 2 & -2 \\ -2 & -2 \end{pmatrix}$	$-f_{13}(-2x)$	$g_{13}(-2x)$
65	2	3	$\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$	$f_{42}(x)$	$g_{42}(x)$
65	3	4	$\begin{pmatrix} 0 & 2 \\ 4 & 0 \end{pmatrix}$	$-f_4(x)$	$g_4(x)$
66	1	2	$\begin{pmatrix} -4 & 2 \\ -4 & -2 \end{pmatrix}$	$f_{13}(-x)$	$g_{13}(-x)$
66	2	3	$\begin{pmatrix} 0 & 2 \\ 4 & 0 \end{pmatrix}$	$f_{42}(x)$	$g_{42}(x)$
66	3	4	$\begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}$	$f_4(x)$	$g_4(x)$
71	1	2	$\begin{pmatrix} 0 & -\frac{1}{2} \\ 1 & 0 \end{pmatrix}$	0	$g_{47}(-2x)$
72	1	2	$\begin{pmatrix} 0 & -1 \\ 4 & 0 \end{pmatrix}$	0	$g_{47}(-x)$
73	1	2	$\begin{pmatrix} 0 & 1 \\ 4 & 0 \end{pmatrix}$	0	$g_{47}(x)$
74	1	2	$\begin{pmatrix} 0 & \frac{1}{2} \\ 1 & 0 \end{pmatrix}$	0	$g_{47}(2x)$
75	1	2	$\begin{pmatrix} 0 & \frac{1}{2} \\ 2 & 0 \end{pmatrix}$	0	$g_{48}(x)$
75	2	3	$\begin{pmatrix} 0 & -1 \\ -2 & 0 \end{pmatrix}$	$f_8(x)$	$g_8(x)$
75	2	4	$\begin{pmatrix} -2 & 0 \\ 4 & -2 \end{pmatrix}$	$2f_{10}(x)$	-2
76	1	2	$\begin{pmatrix} 0 & \frac{1}{2} \\ 1 & 0 \end{pmatrix}$	0	$g_{48}(2x)$
76	2	3	$\begin{pmatrix} 0 & -1 \\ -2 & 0 \end{pmatrix}$	$f_8(x)$	$g_8(x)$
76	2	4	$\begin{pmatrix} -2 & 0 \\ 4 & -2 \end{pmatrix}$	$2f_{10}(x)$	-2
77	1	2	$\begin{pmatrix} 0 & -\frac{1}{2} \\ 1 & 0 \end{pmatrix}$	0	$g_{48}(-2x)$
77	2	3	$\begin{pmatrix} 0 & -1 \\ -2 & 0 \end{pmatrix}$	$f_8(x)$	$g_8(x)$
77	2	4	$\begin{pmatrix} -2 & 0 \\ 4 & -2 \end{pmatrix}$	$2f_{10}(x)$	-2
78	1	2	$\begin{pmatrix} 0 & -\frac{1}{2} \\ 2 & 0 \end{pmatrix}$	0	$g_{48}(-x)$
78	2	3	$\begin{pmatrix} 0 & -1 \\ -2 & 0 \end{pmatrix}$	$f_8(x)$	$g_8(x)$
78	2	4	$\begin{pmatrix} -2 & 0 \\ 4 & -2 \end{pmatrix}$	$2f_{10}(x)$	-2
79	1	2	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$	$f_{30}(x)$	$g_{30}(x)$
80	1	2	$\begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}$	$f_{28}(x)$	$g_{28}(x)$
81	1	2	$\begin{pmatrix} 0 & \frac{1}{2} \\ 1 & 0 \end{pmatrix}$	0	$g_2(x)$
82	1	2	$\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}$	$f_{36}(2x)$	$g_{36}(2x)$

TABLE 3. (continued)

n	i	j	$\overline{\alpha}$	s_1	s_2
82	2	3	$\begin{pmatrix}0&-2\\2&0\end{pmatrix}$	$f_{43}(x)$	$g_{43}(x)$
82	3	4	$\begin{pmatrix}0&-1\\2&0\end{pmatrix}$	$f_{35}(x)$	$g_{35}(x)$
83	1	2	$\begin{pmatrix}-4&-2\\-4&2\end{pmatrix}$	$f_{36}(x)$	$g_{36}(x)$
83	2	3	$\begin{pmatrix}0&-1\\1&0\end{pmatrix}$	$f_{43}(x)$	$g_{43}(x)$
83	3	4	$\begin{pmatrix}0&-2\\4&0\end{pmatrix}$	$f_{35}(x)$	$g_{35}(x)$
84	1	2	$\begin{pmatrix}4&-2\\-4&-2\end{pmatrix}$	$-f_{36}(-x)$	$g_{36}(-x)$
84	2	3	$\begin{pmatrix}0&-1\\1&0\end{pmatrix}$	$f_{43}(x)$	$g_{43}(x)$
84	3	4	$\begin{pmatrix}0&-2\\4&0\end{pmatrix}$	$f_{35}(x)$	$g_{35}(x)$
85	1	2	$\begin{pmatrix}1&-1\\-1&-1\end{pmatrix}$	$-f_{36}(-2x)$	$g_{36}(-2x)$
85	2	3	$\begin{pmatrix}0&-2\\2&0\end{pmatrix}$	$f_{43}(x)$	$g_{43}(x)$
85	3	4	$\begin{pmatrix}0&-1\\2&0\end{pmatrix}$	$f_{35}(x)$	$g_{35}(x)$
86	1	2	$\begin{pmatrix}-4&-2\\0&2\end{pmatrix}$	$-f_9(-x)$	$g_9(-x)$
86	1	4	$\begin{pmatrix}-1&\frac{1}{2}\\-4&-1\end{pmatrix}$	$f_{24}(x)$	$g_{24}(x)$
86	3	4	$\begin{pmatrix}\frac{1}{2}&-\frac{1}{2}\\0&-1\end{pmatrix}$	$2f_9(2x)$	$4g_9(2x)$
87	1	4	$\begin{pmatrix}2&2\\-2&2\end{pmatrix}$	$f_5(2x)$	$g_5(2x)$
87	2	3	$\begin{pmatrix}0&1\\4&0\end{pmatrix}$	$-2f_4(x)$	$4g_4(x)$
87	2	4	$\begin{pmatrix}4&0\\0&-2\end{pmatrix}$	$f_{18}(x)$	$g_{18}(x)$
88	1	4	$\begin{pmatrix}4&2\\-4&2\end{pmatrix}$	$f_5(x)$	$g_5(x)$
88	2	3	$\begin{pmatrix}0&1\\4&0\end{pmatrix}$	$-2f_4(x)$	$4g_4(x)$
88	2	4	$\begin{pmatrix}4&0\\0&-2\end{pmatrix}$	$f_{18}(x)$	$g_{18}(x)$
89	1	4	$\begin{pmatrix}4&-2\\-4&-2\end{pmatrix}$	$f_5(-x)$	$g_5(-x)$
89	2	3	$\begin{pmatrix}0&1\\4&0\end{pmatrix}$	$-2f_4(x)$	$4g_4(x)$
89	2	4	$\begin{pmatrix}4&0\\0&-2\end{pmatrix}$	$f_{18}(x)$	$g_{18}(x)$
90	1	4	$\begin{pmatrix}-2&2\\-2&-2\end{pmatrix}$	$-f_5(-2x)$	$g_5(-2x)$
90	2	3	$\begin{pmatrix}0&1\\4&0\end{pmatrix}$	$-2f_4(x)$	$4g_4(x)$
90	2	4	$\begin{pmatrix}4&0\\0&-2\end{pmatrix}$	$f_{18}(x)$	$g_{18}(x)$
93	1	2	$\begin{pmatrix}4&-2\\0&-2\end{pmatrix}$	$f_9(x)$	$g_9(x)$
93	1	4	$\begin{pmatrix}-1&1\\4&-1\end{pmatrix}$	$f_{23}(x)$	$g_{23}(x)$
93	3	4	$\begin{pmatrix}-\frac{1}{2}&-\frac{1}{2}\\0&1\end{pmatrix}$	$-2f_9(-2x)$	$4g_9(-2x)$
94	1	3	$\begin{pmatrix}1&1\\-1&1\end{pmatrix}$	$f_5(2x)$	$g_5(2x)$
94	2	4	$\begin{pmatrix}0&1\\-1&0\end{pmatrix}$	$f_3(x)$	$g_3(x)$
94	3	4	$\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$	$f_{17}(x)$	$g_{17}(x)$
95	1	3	$\begin{pmatrix}4&2\\-4&2\end{pmatrix}$	$f_5(x)$	$g_5(x)$
95	2	4	$\begin{pmatrix}0&1\\-1&0\end{pmatrix}$	$f_3(x)$	$g_3(x)$
95	3	4	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$	$-f_{17}(x)$	$g_{17}(x)$
96	1	3	$\begin{pmatrix}4&-2\\-4&-2\end{pmatrix}$	$f_5(-x)$	$g_5(-x)$
96	2	4	$\begin{pmatrix}0&-1\\-1&0\end{pmatrix}$	$-f_3(x)$	$g_3(x)$
96	3	4	$\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$	$f_{17}(x)$	$g_{17}(x)$
97	1	3	$\begin{pmatrix}-1&1\\-1&-1\end{pmatrix}$	$-f_5(-2x)$	$g_5(-2x)$
97	2	4	$\begin{pmatrix}0&-1\\-1&0\end{pmatrix}$	$-f_3(x)$	$g_3(x)$
97	3	4	$\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$	$f_{17}(x)$	$g_{17}(x)$
98	1	2	$\begin{pmatrix}0&\frac{1}{4}\\1&0\end{pmatrix}$	0	$4g_2(x)$
99	1	2	$\begin{pmatrix}0&1\\4&0\end{pmatrix}$	0	$4g_2(x)$
100	1	3	$\begin{pmatrix}1&0\\-1&-1\end{pmatrix}$	$f_{32}(x)$	$g_{32}(x)$
101	1	2	$\begin{pmatrix}1&0\\-1&1\end{pmatrix}$	$f_{32}(-x)$	$g_{32}(-x)$
102	1	2	$\begin{pmatrix}0&\frac{1}{2}\\1&0\end{pmatrix}$	$f_{27}(x)$	$g_{27}(x)$
103	1	2	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$f_{29}(x)$	$g_{29}(x)$
104	1	2	$\begin{pmatrix}-1&0\\0&-1\end{pmatrix}$	$f_{46}(x)$	-2
105	1	2	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$	$-f_{46}(x)$	-2
106	1	3	$\begin{pmatrix}-1&0\\-3&-1\end{pmatrix}$	$f_7(x)$	$g_7(x)$
107	1	3	$\begin{pmatrix}-1&0\\-3&-1\end{pmatrix}$	$f_7(x)$	$g_7(x)$
110	1	2	$\begin{pmatrix}-1&0\\1&1\end{pmatrix}$	$f_{19}(x)$	$g_{19}(x)$
110	1	4	$\begin{pmatrix}1&0\\1&1\end{pmatrix}$	$f_{21}(x)$	$g_{21}(x)$
111	1	3	$\begin{pmatrix}-1&0\\1&1\end{pmatrix}$	$f_{19}(x)$	$g_{19}(x)$
111	1	5	$\begin{pmatrix}1&0\\1&1\end{pmatrix}$	$f_{21}(x)$	$g_{21}(x)$
112	1	2	$\begin{pmatrix}1&-\frac{1}{2}\\-1&1\end{pmatrix}$	$f_{11}(x)$	$2g_{11}(x)$
113	1	2	$\begin{pmatrix}1&-\frac{1}{2}\\-1&1\end{pmatrix}$	$f_{11}(x)$	$2g_{11}(x)$
118	1	2	$\begin{pmatrix}0&2\\4&0\end{pmatrix}$	$f_{40}(x)$	$g_{40}(x)$
119	1	2	$\begin{pmatrix}0&1\\2&0\end{pmatrix}$	$f_{40}(x)$	$g_{40}(x)$
120	1	2	$\begin{pmatrix}0&1\\2&0\end{pmatrix}$	$f_{40}(x)$	$g_{40}(x)$

TABLE 3. (continued)

n	i	j	$\overline{\alpha}$	s_1	s_2
121	1	2	$\begin{pmatrix}0&2\\4&0\end{pmatrix}$	$f_{40}(x)$	$g_{40}(x)$
126	2	3	$\begin{pmatrix}-1&-1\\-1&1\end{pmatrix}$	$f_{37}(x)$	$g_{37}(x)$
127	1	4	$\begin{pmatrix}-1&-1\\-1&1\end{pmatrix}$	$f_{37}(x)$	$g_{37}(x)$
130	1	2	$\begin{pmatrix}2&2\\2&0\end{pmatrix}$	$f_{39}(x)$	$g_{39}(x)$
131	1	2	$\begin{pmatrix}2&2\\2&0\end{pmatrix}$	$f_{39}(x)$	$g_{39}(x)$
132	1	2	$\begin{pmatrix}-2&2\\0&-2\end{pmatrix}$	$f_{34}(x)$	$g_{34}(x)$
133	1	3	$\begin{pmatrix}-2&2\\0&-2\end{pmatrix}$	$f_{34}(x)$	$g_{34}(x)$
134	1	2	$\begin{pmatrix}1&0\\1&-2\end{pmatrix}$	$f_{48}(2x)$	$g_{11}(2x)$
135	1	2	$\begin{pmatrix}-1&1\\0&-1\end{pmatrix}$	$f_{48}(2x)$	$g_{11}(2x)$
140	1	2	$\begin{pmatrix}2&0\\0&-2\end{pmatrix}$	$f_{25}(x)$	$g_{25}(x)$
141	1	2	$\begin{pmatrix}2&0\\0&-2\end{pmatrix}$	$f_{25}(x)$	$g_{25}(x)$
142	1	2	$\begin{pmatrix}-2&0\\0&-2\end{pmatrix}$	$-f_{25}(x)$	$g_{25}(x)$
143	1	2	$\begin{pmatrix}2&0\\0&-2\end{pmatrix}$	$f_{25}(x)$	$g_{25}(x)$
144	2	4	$\begin{pmatrix}-1&1\\0&-1\end{pmatrix}$	$f_{48}(2x)$	$g_{11}(2x)$
145	2	3	$\begin{pmatrix}1&0\\1&-2\end{pmatrix}$	$f_{48}(2x)$	$g_{11}(2x)$
150	1	2	$\begin{pmatrix}2&2\\2&0\end{pmatrix}$	$f_{10}(x)$	$g_{10}(x)$
151	1	2	$\begin{pmatrix}1&1\\1&0\end{pmatrix}$	$f_{10}(x)$	$g_{10}(x)$
152	1	2	$\begin{pmatrix}1&1\\1&0\end{pmatrix}$	$f_{10}(x)$	$g_{10}(x)$
153	1	2	$\begin{pmatrix}2&2\\2&0\end{pmatrix}$	$f_{10}(x)$	$g_{10}(x)$
154	2	3	$\begin{pmatrix}-1&0\\2&-1\end{pmatrix}$	$f_{12}(x)$	$g_{12}(x)$
155	1	3	$\begin{pmatrix}1&1\\-1&1\end{pmatrix}$	$f_{12}(x)$	$g_{12}(x)$
160	1	3	$\begin{pmatrix}-2&0\\2&-2\end{pmatrix}$	$f_{47}(x)$	-2
161	1	3	$\begin{pmatrix}-1&0\\1&-1\end{pmatrix}$	$f_{47}(x)$	-2
162	1	3	$\begin{pmatrix}-1&0\\1&-1\end{pmatrix}$	$f_{47}(x)$	-2
163	1	3	$\begin{pmatrix}-2&0\\2&-2\end{pmatrix}$	$f_{47}(x)$	-2

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTH AFRICA, P. O. BOX 392, PRETORIA, 0003, SOUTH AFRICA

Current address: Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803-4918

E-mail address: `wamelen@math.lsu.edu`